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An equation of anisotropic friction with sliding path curvature effects

Alfred Zmitrowicz*

Institute of Fluid-Flow Machinery, Polish Academy of Sciences, ul.J.Fiszera 14, PL-80-952 Gdańsk, Poland

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Abstract

Anisotropy and inhomogeneity of dry friction can induce a dependence of friction force on a sliding path curvature. The objective of this study is to extend mathematical models of anisotropic friction by including the sliding path curvature effects. Due to this, a set of independent variables of the friction force constitutive equation is extended, and a derivative of the sliding velocity unit vector is taken into account. The friction description is investigated in a general case and in a particular when non-homogeneous friction properties form concentric circles in a contact surface. It has been found that: the friction constitutive equation and its variables satisfy the axiom of objectivity; the Second Law of Thermodynamics restricts components of tensors in the constitutive equation; there are radial and concentric circular privileged sliding directions. Non-homogeneous friction in the form of concentric circles has the global axial symmetry, while two tensors in the constitutive equation are defined locally and they have orthotropic and anisotropic properties, respectively. The sliding path curvature can induce positive and negative additional friction. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Modern technology stimulates developments of mathematical models of friction, by requiring a more precise description of the frictional resistance to be considered in numerical tools (FEM). To provide a more realistic model of friction, there is a need to extend friction descriptions by including non-homogeneity and anisotropy, since a certain amount of frictional inhomogeneity and anisotropy is present in component parts of mechanical systems fabricated from material with complex microstructure (composites, polymers, ceramics, etc.). In order to design or develop good machinery component parts it is necessary to characterize and understand the friction and wear

* Fax: 00 48 58 41 61 44; E-mail: azmit@imppan.imp.pg.gda.pl

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behavior of materials with complex microstructure. The problem gives motivations for investigations of anisotropy and non-homogeneous phenomena of friction, wear and frictional heat.

In general, anisotropy of friction and wear results from the roughness anisotropy of contacting surfaces and anisotropy of mechanical properties of crystals, fibrous and laminated composites, polymers, ceramics and other materials with microstructure (e.g. layer–lattice materials).

In the past scientists such as Coulomb (1785) and Morin (1833–1836), observed in their friction experiments on wooden surfaces, that friction coefficients were dependent on whether sliding took place parallel to or perpendicular to the wood fibers. Recent experiments of anisotropy friction in wood and composites have been carried out by Curnier (1996). The literature devoted to experimental and theoretical investigations on anisotropic friction, wear and frictional heat was presented by Zmitrowicz (1992a, b, 1993a, b, 1995a, b).

Polytetrafluoroethylene (PTFE) and high density polyethylene (HDPE) polymers are sensitive to the orientation of their molecular chains with respect to the sliding direction. In pin-on-disc tests, Briscoe and Stolarski (1979, 1985, 1986, 1991) observed that a rate of wear was also a function of a curvature of sliding trajectories which the polymer pin described on the disc surface. By changing a radius of a circular path, different wear rates were observed. An angular velocity of rotation and the radius of rotation were varied, while the normal pressure and the linear sliding velocity were maintained at constant values for each test. Therefore, sliding conditions of the pin against the disc did not change. Maximum wear rate was for large radius of the path and significant reduction in wear rate was when the radius approached the radius of the pin. There was a factor-of-3.5 change in values of the wear rates. Measurements of the frictional force indicated that radii effects on the friction were less pronounced in comparison with the wear rates.

In the opinion of Briscoe and Stolarski (1979, 1985, 1991), the investigated polymers wear by a creation of 'transferred films highly oriented' in the direction of sliding, and the sliding occurs between oriented fibrils. Briscoe and Stolarski (1979, 1985, 1991) thought that an increase of curvature of trajectories, and the resulting increase of reorientation of molecular chains within the contact area are responsible for the observed behavior of the polymers.

In the present analysis, we postulate that the influence of the sliding path curvature on friction can result from anisotropy and inhomogeneity of sliding surface friction. The phenomena of abrasive wear and dry friction are usually treated as inseparable, this way the postulate deals with wear as well.

Physical properties of solids and their surfaces are very often non-homogeneous and they can form different field singularities in a contact between two bodies (in geometric terms: series of radii, concentric circles and ellipses, spirals, etc.). Inhomogeneity refers to physical properties of materials (e.g. wood, crystals, composites) or to specific techniques of material manufacture and finishing surfaces, see Fig. 1. For example, microphotographs carried out by Tolansky (1968) show growth spirals on a crystal face of silicon carbide.

Non-homogeneous physical properties of solids and their surfaces are the causes of the non-homogeneous anisotropy friction. This is the case where anisotropy friction depends on the position of a contact point with respect to a singular field center. Then, the frictional resistance follows on the one hand from the physical properties of the surface, on the other hand it can additionally depend on a sliding trajectory in this surface. Different resistances to sliding can occur for rectilinear or curved trajectories, for a circular path with small curvature radius or the circular path with great curvature radius, etc. In the contact surface with complex properties besides rectilinear

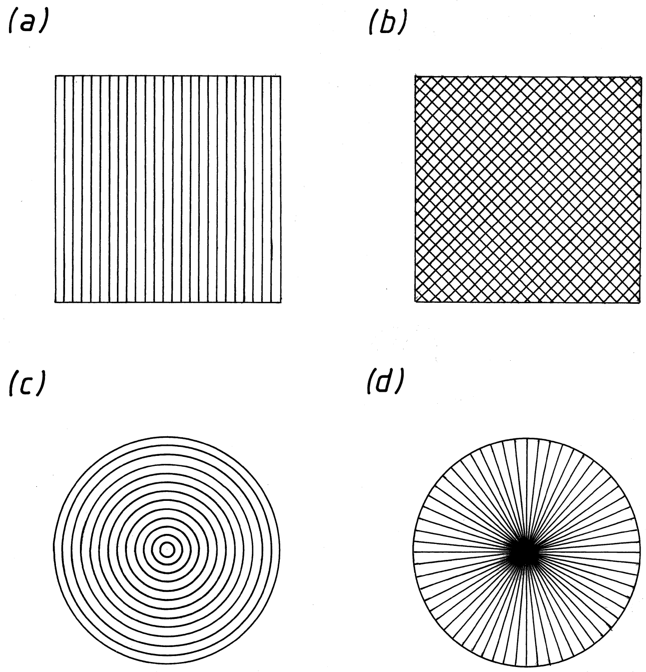

Fig. 1. Four types of conventional machined patterns in a surface: (a) rectilinear and parallel; (b) crossed in two slant directions; (c) circular relative to the center of the surface; (d) radial relative to the center of the surface.

privileged friction directions also curved privileged directions can exist (concentric circles, ellipses, spirals etc.). These facts should be included in the mathematical formulation of the friction law. Former models of anisotropy friction paid no attention to the anisotropic non-homogeneous friction and to the sliding path, see: Moszyński (1951), Moreau (1970, 1974), Michałowski and Mróz (1978), Ziegler (1981), Aleksandrovich et al. (1985), Goyal et al. (1991), Mróz and Stupkiewicz (1992, 1994), He and Curnier (1993), Telega (1995) and Buczkowski and Kleiber (1997).

The objective of this study is to present our first research results towards the contribution of the sliding path curvature effects in mathematical models of anisotropic friction and to give a suitable framework allowing a rational and unified formulation of non-homogeneous anisotropic friction. The influence of the sliding path on friction is included in the friction equation by an extension of a set of independent variables taking into account a derivative of the sliding velocity unit vector. Thermodynamical limitations, symmetry properties and privileged sliding directions are discussed in detail.

2. Assumptions

A trajectory or path of a moving material point in the surface R^2 in the range of time $I = (t_0, t_e)$ is defined by an image set of the set I and the following mapping

$$I \ni t \rightarrow \mathbf{x}(t) \in R^2, \tag{1}$$

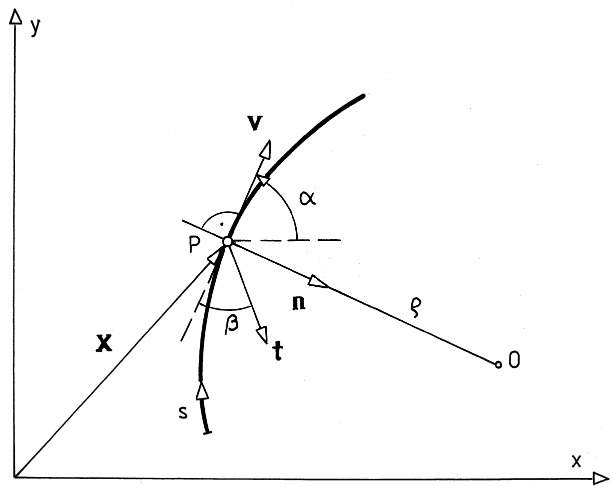

Fig. 2. Sliding trajectory of a material point in a plane.

where, $\mathbf{x}$ is a radius vector, t is time. With the aid of the mapping (1) we can introduce an arc length parameter s and a one-dimensional parameterization of the plane curve (motion trajectory), Fig. 2. The following relation exists between the arc length parameter s and the time t

$$s(t) = \int_{t_0}^{t} |\mathbf{V}(\tau)|\mathrm{d}\tau, \tag{2}$$

$$I \ni t \to s(t) \in R^1, \tag{3}$$

where $\mathbf{V}$ is a velocity vector. A number defined by (2) and (3) is called a way passed by the material point from the instant t_0 to the instant t. The mappings (1) and (3) are the differentiable functions.

From the definition of the velocity vector it follows that this vector is always tangent to a curve representing the motion path

$$\mathbf{V} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \mathbf{v}V, \tag{4}$$

where, V is a velocity value

$$V \equiv |\mathbf{V}| = \frac{\mathrm{d}s}{\mathrm{d}t}, \tag{5}$$

and $\mathbf{v}$ denotes a unit vector tangent to the trajectory ($\mathbf{v}\cdot\mathbf{v} = 1$), i.e.

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}. \tag{6}$$

According to the Frenet–Serret first formula, we have

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s} = \frac{\mathbf{n}}{\rho}, \tag{7}$$

where, $\mathbf{n}$ is a unit vector normal to the path ($\mathbf{n}\cdot\mathbf{v} = 0, \mathbf{n}\cdot\mathbf{n} = 1$), ρ is a path curvature radius

$$\rho = \left|\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}\right|^{-1}. \tag{8}$$

$1/\rho$ is called the curvature. The vector $\mathrm{d}\mathbf{v}/\mathrm{d}s$ at any point of the path is normal to the unit vector $\mathbf{v}$ or it vanishes. The length of the vector $\mathrm{d}\mathbf{v}/\mathrm{d}s$ is equal to the inverse of the curvature radius. Thus, $\mathrm{d}\mathbf{v}/\mathrm{d}s$ is not a unit vector.

The parameterization s determines uniquely the pair of vectors $\mathbf{v}$ and $\mathbf{n}$, see (6) and (7). Both definitions are always valid, i.e. for any instant of time and any place of the motion trajectory. Unit vectors $\mathbf{v}$ and $\mathbf{n}$ are orthogonal at any point of the curve, Fig. 2. The unit vector $\mathbf{v}$ is directed in the motion direction, and $\mathbf{n}$ is always directed into the curvature center. During the motion along the curve the unit vectors change their directions (with respect to an observer or the reference system) but they remain orthogonal.

Usually it has been assumed, that the friction force vector $\mathbf{t}$ at the point $\mathbf{x}$ of the contact surface has the same value for all sliding trajectories which pass through $\mathbf{x}$ and have the sliding velocity unit vector $\mathbf{v}$. In other words, the friction force vector depends on the velocity direction only through the sliding velocity unit vector $\mathbf{v}$ at $\mathbf{x}$, it does not depend on other sliding trajectory parameters. We postulate that the friction force vector at the given point depends on the sliding velocity direction $\mathbf{v}$ and on the sliding path curvature. Therefore, we extend a set of independent variables of a friction force equation taking into account the derivative of the sliding velocity unit vector i.e. $\mathrm{d}\mathbf{v}/\mathrm{d}s$.

Let us replace the friction force vector formulation of the form

$$\mathbf{t} = -Nf(\mathbf{v}) \tag{9}$$

by the following definitions

$$\mathbf{t} = -Nf\left(\mathbf{v}, \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}\right), \tag{10}$$

$$\mathbf{t} = -Nf\left(\mathbf{v}, \frac{\mathbf{n}}{\rho}\right), \tag{11}$$

where, $N \geqslant 0$ is a normal pressure (Zmitrowicz, 1995). The sum of two monomials (i.e. single-term polynomials) defines a simple form of the function (11), i.e.

$$\mathbf{t} = -N\left(\mathbf{C}\mathbf{v} + \mathbf{E}\frac{\mathbf{n}}{\rho}\right). \tag{12}$$

Subsequent terms in the equation (12) play the following roles: $\mathbf{t}$ is the response; N, $\mathbf{v}$ and $\mathbf{n}/\rho$ are causes; $\mathbf{C}$ and $\mathbf{E}$ are equation coefficients (parametric tensors). The second order tensor $\mathbf{C}$ defines friction resulting from physical properties of the contact surface i.e. anisotropy and inhomogeneity. Tensor $\mathbf{E}$ includes the effects associated with the sliding motion i.e. constraints imposed on the sliding motion. The well-known friction postulates of Amontons and Coulomb are accepted in the formulation (12).

The friction force (12) can be expressed as a sum of two components, i.e.

$$\mathbf{t} = \mathbf{t}_0 + \mathbf{t}_\rho, \tag{13}$$

where

$$\mathbf{t}_0 = -N\mathbf{C}\mathbf{v}, \tag{14}$$

$$\mathbf{t}_\rho = -N\mathbf{E}\frac{\mathbf{n}}{\rho} \equiv -N\mathbf{E}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}. \tag{15}$$

The component $\mathbf{t}_0$ does not depend on the sliding path curvature, the component $\mathbf{t}_\rho$ depends on the curvature.

The vector $\mathrm{d}\mathbf{v}/\mathrm{d}s$ introduces effects caused by the sliding path curvature. If the relative motion trajectory is a curve at the given contact point, then the contribution of $\mathrm{d}\mathbf{v}/\mathrm{d}s$ can be taken into account (exactly, if the curvature radius ρ has a finite value). If the sliding path is a straight line, then the curvature vanishes ($1/\rho = 0$), and the dependence between the friction force $\mathbf{t}$ and $\mathrm{d}\mathbf{v}/\mathrm{d}s$ vanishes.

In the light of the formulation (12), in the singular case ($\rho = 0$), the friction component $\mathbf{t}_\rho$ tends to infinity, so the friction force $\mathbf{t}$ is not determined, and it must be excluded from the considerations. However, the singular case ($\rho = 0$) is physically meaningless. It can be explained with the aid of the following example. Taking into account concentric circular trajectories, one can reduce radii of the circles from a finite value up to zero. In the singular case, the curvature radius is equal to zero, and the circular sliding trajectory reduces to a single point. If the circular trajectory reduces to the single point ($\rho = 0$), then tangent ($\mathbf{v}$) and normal ($\mathbf{n}$) directions to the trajectory cannot be distinguished uniquely, there are no sliding and no dynamic friction.

Very different case takes place, if two bodies are instantly at the contact at a single point and a relative velocity has a finite value and a definite direction. Then, the friction force is completely described by the unit vector of the relative velocity and the normal pressure. In this case, the component $\mathbf{t}_\rho$ can be neglected, since the sliding trajectory at the initial instant can be assumed to be a straight line (i.e. $\rho = \infty$).

The friction force component $\mathbf{t}_\rho$ has the following values for different curvature radii

$$\mathbf{t}_\rho \begin{cases} = \mathbf{0}, & \text{for } \rho = \infty \\ \neq \mathbf{0}, & \text{for } 0 < \rho < \infty \\ = \infty, & \text{for } \rho = 0 \end{cases} \tag{16}$$

Friction which depends on the direction of sliding is called anisotropic friction. A deviation in the friction force from the direction of sliding and a dependence of the friction magnitude on the sliding direction are typical features of contacts with frictional anisotropy. Anisotropic friction forces vary from the assigned sliding direction, Fig. 2. Isotropic friction forces always act in a direction opposite to that of the sliding velocity.

The anisotropic friction coefficient μ_α and the angle β of friction force inclination (Fig. 2) for any sliding direction can be obtained from the following relations

$$\mu_\alpha = N^{-1}|\mathbf{t}|, \tag{17}$$

$$\sin\beta = \frac{\mathbf{t}\cdot\mathbf{n}}{|\mathbf{t}|}. \tag{18}$$

Coefficients of the friction force components collinear with the sliding direction and normal to the sliding direction are given by

$$\mu_\alpha^{\parallel} = -N^{-1}\mathbf{t}\cdot\mathbf{v}, \tag{19}$$

$$\mu_\alpha^{\perp} = N^{-1}\mathbf{t}\cdot\mathbf{n}. \tag{20}$$

It happens very often that fields of pressure and relative velocity are inhomogeneous in a contact area. In spite of that a local constitutive law of friction should be independent of that whether its independent variables deal with homogeneous or non-homogeneous pressure and velocity fields. A different situation takes place if physical properties of the surface are inhomogeneous.

In some cases, physical reasons of anisotropy friction (wood and composite fibers, machining marks, etc.) form non-homogeneous fields in the surface, and in geometric terms they cannot be represented as a series of parallel and equidistant straight lines but as series of radii, concentric circles and ellipses, spirals, etc. For example, Fig. 1 shows four types of common directions of lay in a surface.

Let us consider non-homogeneous friction properties which form concentric circles in the contact surface, Fig. 1(c). The tensor $\mathbf{C}$ defines anisotropic and non-homogeneous friction properties of this surface, and it gives a local description of friction (i.e. the description at the given point of the contact). The tensor has the following form

$$\mathbf{C} = \mu_1\mathbf{k}_1\otimes\mathbf{k}_1 + \mu_2\mathbf{k}_2\otimes\mathbf{k}_2. \tag{21}$$

$\mathbf{k}_1$ and μ_1 are associated with the tangent to the concentric circles in the case of $\mathbf{k}_1$ and with the sliding along the circles in the case of μ_1, it holds at any point of the contact. $\mathbf{k}_2$ and μ_2 are defined for the sliding along the radii of the concentric circles. Therefore, unit vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ form an orthogonal basis at any point of the contact surface (Fig. 3). Friction coefficients μ_1 and μ_2 have constant values, and they define friction along the concentric circles (curved directions) and along the radii (rectilinear directions), respectively.

The tensor $\mathbf{E}$ includes the effects associated with the sliding motion, and it has the following form

$$\mathbf{E} = \eta_1\mathbf{v}\otimes\mathbf{n} + \eta_2\mathbf{n}\otimes\mathbf{n}. \tag{22}$$

Here, η_1 is a coefficient of constraints imposed on the motion in direction tangent to the sliding path, η_2 is a coefficient of constraints imposed on the motion in direction normal to the sliding path. Coefficient η_1 defines the nonconservative force (dissipative type force), whereas coefficient η_2 defines the conservative force (gyroscopic type force). Both coefficients η_1 and η_2 are given in meter units.

The definitions of tensors $\mathbf{C}$ and $\mathbf{E}$ are valid for all sliding trajectories (rectilinear, curved, zig-zag, etc), and for trajectories arbitrary oriented with respect to the concentric circles. If a contact plane has isotropic and homogeneous friction properties, then $\mathbf{C} = \mu\mathbf{1}$, $\mathbf{E} = \mathbf{0}$ and $\mathbf{t} = -\mu N\mathbf{v}$.

After substitution tensor (22) into (15) we obtain

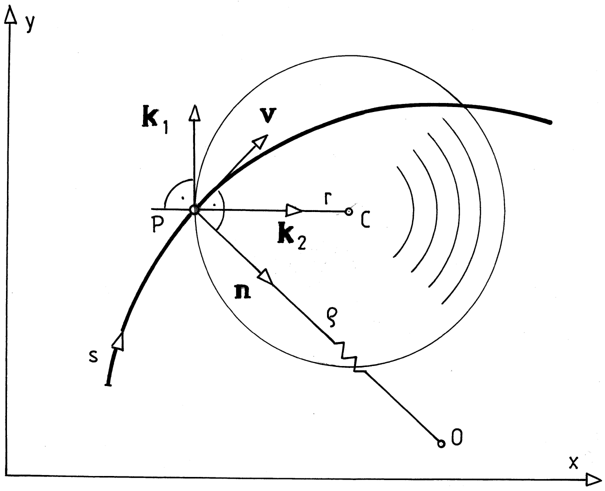

Fig. 3. Sliding trajectory of a material point in a plane with non-homogeneous friction properties which form concentric circles.

$$\mathbf{t}_\rho = -\frac{N\eta_1}{\rho}\mathbf{v} - \frac{N\eta_2}{\rho}\mathbf{n}. \tag{23}$$

The sliding path curvature generates: an additional friction (η_1) and a reaction to constraints normal to the curved sliding path (η_2).

Both terms in (23) depend on the curvature radius ρ. This way, the eqn (12) describes the effects of the sliding path shape on friction.

Tensor bases of **C** and **E** depend on the position in the case of **C** and on the sliding motion in the case of **E**. Therefore, unit vectors of the tensor bases have different orientations at various points of the contact area with respect to the reference system, Fig. 3. The basis $\{\mathbf{k}_1, \mathbf{k}_2\}$ can be transformed to the basis $\{\mathbf{v}, \mathbf{n}\}$ with the aid of the following transformation rule between the unit vectors

$$[\mathbf{k}_1, \mathbf{k}_2]^T = \mathbf{B}[\mathbf{v}, \mathbf{n}]^T. \tag{24}$$

Coefficients of the transformation matrix $\mathbf{B} = [B_{ij}]$, $i, j = 1, 2$ are defined by

$$B_{11} = \mathbf{k}_1 \cdot \mathbf{v}, \quad B_{12} = \mathbf{k}_1 \cdot \mathbf{n}, \tag{25}$$

$$B_{21} = \mathbf{k}_2 \cdot \mathbf{v}, \quad B_{22} = \mathbf{k}_2 \cdot \mathbf{n}. \tag{26}$$

The transformation matrix **B** depends on the reference location. Hence, the tensor (21) takes the following form

$$\mathbf{C} = [\mu_1(B_{11})^2 + \mu_2(B_{21})^2]\mathbf{v} \otimes \mathbf{v} + (\mu_1 B_{11} B_{12} + \mu_2 B_{21} B_{22})(\mathbf{v} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{v})$$
$$+ [\mu_1(B_{12})^2 + \mu_2(B_{22})^2]\mathbf{n} \otimes \mathbf{n}. \tag{27}$$

After substitution (27) into (14), the component $\mathbf{t}_0$ is given by

$$\mathbf{t}_0 = -N\{[\mu_1(B_{11})^2+\mu_2(B_{21})^2]\mathbf{v}+(\mu_1 B_{11}B_{12}+\mu_2 B_{21}B_{22})\mathbf{n}\}. \tag{28}$$

The tensor $\mathbf{C}$ can be defined in the following general form

$$\mathbf{C}(\mathbf{X}) = C_{ij}\mathbf{k}_i \otimes \mathbf{k}_j, \quad i,j = 1,2 \quad C_{ij} = \text{const}, \quad \mathbf{k}_i = \mathbf{k}_i(\mathbf{X}), \tag{29}$$

where, $\{\mathbf{k}_1, \mathbf{k}_2\}$ is a basis of orthogonal or arbitrary unit vectors adequately oriented with respect to field singularities (i.e. with respect to a series of radii, concentric circles and ellipses, spirals, etc.). The unit vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ composing the tensor basis $(\mathbf{k}_i \otimes \mathbf{k}_j)$ have different orientations with respect to the reference system at various points $\mathbf{X}$ of the contact surface. In spite of that the tensor $\mathbf{C}$ defines steady friction properties of the surface. In this study, the tensor coefficients C_{ij} are constant, and $C_{ij} \in R$.

A general form of the tensor $\mathbf{E}$ is as follows

$$\mathbf{E} = E_{kl}\mathbf{e}_k \otimes \mathbf{e}_l, \quad k,l = 1,2 \quad E_{kl} \neq \text{const}, \quad \{\mathbf{e}_1, \mathbf{e}_2\} \equiv \{\mathbf{v}, \mathbf{n}\}. \tag{30}$$

The orthogonal basis $\{\mathbf{v}, \mathbf{n}\}$ is associated with the sliding trajectory, Fig. 3. There is no definition of the tensor $\mathbf{E}$ in the tensor basis formed by $\mathbf{k}_1$ and $\mathbf{k}_2$ since $\mathbf{E}$ does not depend on friction properties of the contact surface, but it depends on the sliding motion represented by the unit vectors $\mathbf{v}$ and $\mathbf{n}$. The tensor coefficients are arbitrary $E_{kl} \in R$, and some of the tensor coefficients E_{kl} can depend on the sliding velocity V. It does not disturb the dry friction definition, since these tensor coefficients, E_{kl} define a gyroscopic type component of the friction force.

3. Properties

As in continuum mechanics the central topic for the constitutive models of anisotropic and non-homogeneous friction are conditions of material objectivity, the Second Law of Thermodynamics, conditions of symmetry and particular representations of the constitutive relations.

The friction equation in the manner described has the following properties.

Property 1. The friction eqn (12) and its variables satisfy the axiom of material objectivity.

According to the axiom of material objectivity it is required that constitutive equations must be form-invariant with respect to translations and rotations of a reference system or an observer. The axiom of objectivity reduces a class of independent variables and forms of the constitutive functionals that may be used for expressing constitutive equations.

At first we investigate the material objectivity of the independent variables in the constitutive equation, i.e. $\mathbf{v}$ and $\mathrm{d}\mathbf{v}/\mathrm{d}s$. Let us assume that a motion of two contacting bodies A and B is defined with the aid of radius vectors $\mathbf{x}_A(\mathbf{X}_A, t)$ and $\mathbf{x}_B(\mathbf{X}_B, t)$, where $\mathbf{X}_A$ and $\mathbf{X}_B$ are particles of the bodies. The so-called equivalent motions we obtain as a result of a rigid rotation and a translation of the reference system, i.e.

$$\tilde{\mathbf{x}}_A(\mathbf{X}_A, t) = \mathbf{R}(t)\mathbf{x}_A(\mathbf{X}_A, t)+\mathbf{b}(t), \tag{31}$$

$$\tilde{\mathbf{x}}_B(\mathbf{X}_B, t) = \mathbf{R}(t)\mathbf{x}_B(\mathbf{X}_B, t)+\mathbf{b}(t), \tag{32}$$

for every rotation tensor $\mathbf{R}(t)$ and every translation vector $\mathbf{b}(t)$. Differentiating (31) and (32) we get equivalent velocities

$$\tilde{\mathbf{V}}_A = \mathbf{R}\dot{\mathbf{x}}_A + \dot{\mathbf{R}}\mathbf{x}_A + \dot{\mathbf{b}}, \tag{33}$$

$$\tilde{\mathbf{V}}_B = \mathbf{R}\dot{\mathbf{x}}_B + \dot{\mathbf{R}}\mathbf{x}_B + \dot{\mathbf{b}}. \tag{34}$$

If particles $\mathbf{X}_A$ and $\mathbf{X}_B$ are in contact at the present instant, then the following relation holds

$$\mathbf{x}_A(\mathbf{X}_A, t) = \mathbf{x}_B(\mathbf{X}_B, t). \tag{35}$$

Taking (33)–(35), an equivalent relative velocity at the contact point is given by

$$\tilde{\mathbf{V}}_{AB} = \tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_B = \mathbf{R}(\dot{\mathbf{x}}_A - \dot{\mathbf{x}}_B) \equiv \mathbf{R}\mathbf{V}_{AB}, \tag{36}$$

where, the relative velocity is defined by

$$\mathbf{V}_{AB} = \dot{\mathbf{x}}_A - \dot{\mathbf{x}}_B. \tag{37}$$

Furthermore, the orthogonal tensor $\mathbf{R}$ does not change a vector length, thus we have

$$\tilde{V}_{AB} = |\tilde{\mathbf{V}}_A - \tilde{\mathbf{V}}_B| = |\mathbf{R}\mathbf{V}_{AB}| = |\mathbf{V}_{AB}| \equiv V_{AB}. \tag{38}$$

Therefore, the sliding velocity unit vector transforms in accordance to the following rule

$$\tilde{\mathbf{v}} = \frac{\tilde{\mathbf{V}}_{AB}}{\tilde{V}_{AB}} = \frac{\mathbf{R}\mathbf{V}_{AB}}{V_{AB}} = \mathbf{R}\mathbf{v}. \tag{39}$$

This implies that the sliding velocity unit vector is the objective vector.

The same transformation rule acts in the case of $\mathrm{d}\mathbf{v}/\mathrm{d}s$, i.e.

$$\frac{\mathrm{d}\tilde{\mathbf{v}}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{R}\mathbf{v}) = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s}\mathbf{v} + \mathbf{R}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s} = \mathbf{R}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}, \tag{40}$$

since the rotation tensor $\mathbf{R}(t)$ transforms rigidly the reference system, and it does not depend on the position (s) in the sliding trajectory

$$\mathbf{R} = \mathbf{R}(t) \rightarrow \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s} = \mathbf{0}. \tag{41}$$

It means that for all sliding trajectories tensor $\mathbf{R}$ is the same. Thus, the independent variable $\mathrm{d}\mathbf{v}/\mathrm{d}s$ is the objective vector.

Adopting the rule proposed by Noll, the material objectivity condition for the friction force (10) has the following form

$$\mathbf{t}\left(\mathbf{R}\mathbf{v}, \mathbf{R}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}, N\right) = \mathbf{R}\mathbf{t}\left(\mathbf{v}, \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}, N\right), \quad \forall \mathbf{R} \in O. \tag{42}$$

where, O is the full orthogonal group and the orthogonal tensor $\mathbf{R}$ has the following properties

$$\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{1} \quad \det \mathbf{R} = \pm 1. \tag{43}$$

The friction force function of the unit vector $\mathbf{v}$, vector $\mathrm{d}\mathbf{v}/\mathrm{d}s$ and scalar N must be form-invariant with respect to arbitrary transformation from the full orthogonal group O. After substitution (12) and (42) we get

$$\mathbf{t}\left(\mathbf{Rv}, \mathbf{R}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}, N\right) = -N\left(\mathbf{CRv}+\mathbf{ER}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}\right) = \mathbf{Rt}\left(\mathbf{v}, \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}, N\right). \tag{44}$$

Notice, that the tensors **C** and **E** in (44) are isotropic. Under a change of observer, anisotropic tensors **C** and **E** transform according to the following rule

$$\tilde{\mathbf{C}} = \mathbf{RCR}^T, \quad \tilde{\mathbf{E}} = \mathbf{RER}^T. \tag{45}$$

It means, that a sliding surface with privileged sliding directions changes its orientation under the superposed rigid rotation **R** of the reference system. The orientation of the privileged sliding directions (frictional anisotropy) with respect to the reference system should be maintained. By virtue of the definition (42), we investigate material objectivity in the case of anisotropic tensors **C** and **E** as follows

$$\mathbf{t}\left(\mathbf{Rv}, \mathbf{R}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}, N\right) = -N\left(\tilde{\mathbf{C}}\mathbf{Rv}+\tilde{\mathbf{E}}\mathbf{R}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}\right) = \mathbf{Rt}\left(\mathbf{v}, \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}, N\right). \tag{46}$$

Property 2. The condition of energy dissipated in friction process restricts components of the tensors **C** and **E**.

From the Second Law of Thermodynamics it follows that a power of the friction force in every case of the frictional contact is non-positive (Zmitrowicz, 1992a, 1995b), i.e.

$$\mathbf{t}\cdot\mathbf{V} \leqslant 0, \quad \forall\mathbf{V}. \tag{47}$$

The friction constitutive relations are assumed to satisfy the dissipation inequality (47) for any motion.

Substituting the friction eqn (12) and (47), we obtain

$$-N\left(\mathbf{Cv}+\mathbf{E}\frac{\mathbf{n}}{\rho}\right)\cdot\mathbf{V} \leqslant 0. \tag{48}$$

Taking into account that $N \geqslant 0$ and $V > 0$, the inequality (48) reduces to the condition as follows

$$\mathbf{v}^T\mathbf{Cv}+\mathbf{v}^T\mathbf{En}\frac{1}{\rho} \geqslant 0, \quad \forall\mathbf{V}. \tag{49}$$

Let us consider $\rho > 0$ and the following restrictions for the tensors **C** and **E**

$$\mathbf{v}^T\mathbf{Cv} \geqslant 0, \quad \mathbf{v}^T\mathbf{En} \geqslant 0. \tag{50}$$

Then the inequality (49) is satisfied for every **V** and every positive radius of curvature $\rho \in R^+$, i.e. for every sliding direction and for every sliding trajectory with positive curvature.

Let us assume that $\rho > 0$ and

$$\mathbf{v}^T\mathbf{Cv} > 0, \quad \mathbf{v}^T\mathbf{En} \leqslant 0 \tag{51}$$

then the inequality (49) is satisfied for every **V** and for some trajectories, i.e. for some values of the positive curvature radii

$$\rho \geqslant -\frac{\mathbf{v}^T\mathbf{E}\mathbf{n}}{\mathbf{v}^T\mathbf{C}\mathbf{v}}. \tag{52}$$

If the radius of curvature is negative $\rho < 0$ and the restrictions (50) are taken into account, then the inequality (49) is satisfied for the following radii of curvature

$$\rho \leqslant -\frac{\mathbf{v}^T\mathbf{E}\mathbf{n}}{\mathbf{v}^T\mathbf{C}\mathbf{v}}. \tag{53}$$

Taking $\rho < 0$ and the restrictions (51), the dissipation inequality (48) holds for every negative radius of curvature $\rho \in R^-$.

We do not consider the case $\mathbf{v}^T\mathbf{C}\mathbf{v} < 0$, since in particular cases, i.e. for rectilinear trajectories ($\rho = \infty$) and for homogeneous friction, the inequality (49) reduces always to the restriction of the form $\mathbf{v}^T\mathbf{C}\mathbf{v} \geqslant 0$.

The $\pm$ sign of the radius ρ corresponds to the choice of orientation. We apply the sign plus for radii of the concentric circles.

Property 3. **C** and **E** obey the rules of transformations for tensors.

In general, after transformation to a new reference system, the tensor **C** takes the following form

$$\tilde{\mathbf{C}} = C^{IJ}\tilde{\mathbf{k}}_I \otimes \tilde{\mathbf{k}}_J, \quad I, J = 1, 2. \tag{54}$$

Let a rule of transformation of the tensor basis unit vectors be as follows

$$\mathbf{k}_i = A_i^I\tilde{\mathbf{k}}_I, \quad \mathbf{k}_j = A_j^J\tilde{\mathbf{k}}_J, \quad i, j = 1, 2 \tag{55}$$

where, A_i^I and A_j^J are coefficients of the transformation. Then, we get

$$\mathbf{C} = C^{ij}\mathbf{k}_i \otimes \mathbf{k}_j = C^{ij}A_i^I A_j^J \tilde{\mathbf{k}}_I \otimes \tilde{\mathbf{k}}_J, \tag{56}$$

and

$$C^{IJ} = C^{ij}A_i^I A_j^J. \tag{57}$$

The tensor **E** has the same property. Let the rule of transformation of the basis unit vectors be given by

$$\mathbf{e}_k = D_k^K\tilde{\mathbf{e}}_K, \quad \mathbf{e}_l = D_l^L\tilde{\mathbf{e}}_L, \quad k, l, K, L = 1, 2 \tag{58}$$

where, D_k^K and D_l^L are transformation coefficients. Then, we have

$$\mathbf{E} = E^{kl}\mathbf{e}_k \otimes \mathbf{e}_l = E^{kl}D_k^K D_l^L \tilde{\mathbf{e}}_K \otimes \tilde{\mathbf{e}}_L = E^{KL}\tilde{\mathbf{e}}_K \otimes \tilde{\mathbf{e}}_L = \tilde{\mathbf{E}}, \tag{59}$$

$$E^{KL} = E^{kl}D_k^K D_l^L. \tag{60}$$

We conclude that $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{E}}$ can be calculated knowing C^{ij}, E^{kl} and the rule of transformation from the first to the second reference system. Objects obeying the rules of transformations (56) and (59) are called tensors.

Property 4. If non-homogeneous friction properties are in the form of concentric circles, and if the constraints normal to the curved sliding trajectory are neglected ($\eta_2 = 0$), then there are two principal directions of friction, i.e. radial and tangent to the concentric circles.

The sliding along the so-called principal directions has the following property: the sliding velocity unit vector **v** and the friction force vector **t** are collinear vectors. In the general case, **v** and **t** are not collinear.

The friction force vector expressed by (12) is related to the unit vector **v** indicating the principal direction at the contact by

$$-N\left(\mathbf{C}\mathbf{v}+\mathbf{E}\frac{\mathbf{n}}{\rho}\right) = -N\lambda\mathbf{v} \tag{61}$$

where, λ is a positive number. After substituting into (61) the tensor **E** of the form (22) and assuming $\eta_2 = 0$, we obtain

$$\left(\mathbf{B}\mathbf{C}\mathbf{B}^T+\frac{\eta_1}{\rho}\mathbf{1}\right)\mathbf{v} = \lambda\mathbf{v}, \tag{62}$$

where, **B** is the transformation matrix of the form (25) and (26), the second-order unit tensor is given by $\mathbf{1} = \mathbf{v}\otimes\mathbf{v}+\mathbf{n}\otimes\mathbf{n}$.

Let us consider rectilinear radial sliding directions. Taking the following relations

$$\mathbf{k}_1 = \mathbf{n}, \quad \mathbf{k}_2 = -\mathbf{v}, \quad \rho = \infty, \tag{63}$$

$$[\mathbf{B}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{64}$$

$$[\mathbf{B}\mathbf{C}\mathbf{B}^T] = \begin{bmatrix} \mu_2 & 0 \\ 0 & \mu_1 \end{bmatrix}, \tag{65}$$

the equation of principal directions (62) reduces to the following form

$$\mu_2\mathbf{v} = \lambda\mathbf{v}, \tag{66}$$

where, $\lambda = \mu_2$. Here, λ does not depend on the position with respect to the center of concentric circles. Therefore, every radial direction is the principal direction of friction. For sliding along any radius and for any distance from the center, the friction coefficient has constant value ($\mu_\alpha = \mu_2$) and the inclination angle of the friction force is equal to zero ($\beta = 0$).

Let the sliding trajectory be a circle of the radius $\rho = r$ attached to the center of concentric circles. Using the following relations

$$\mathbf{k}_1 = \mathbf{v}, \quad \mathbf{k}_2 = \mathbf{n}, \quad \rho = r, \tag{67}$$

$$\mathbf{B} = \mathbf{1}, \tag{68}$$

$$[\mathbf{B}\mathbf{C}\mathbf{B}^T] = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \tag{69}$$

the eqn (62) takes the following form

$$\left(\mu_1 + \frac{\eta_1}{r}\right)\mathbf{v} = \lambda\mathbf{v}, \tag{70}$$

where, $\lambda = \mu_1 + \eta_1/r$ for any direction $\mathbf{v}$ tangent to the concentric circles. Here, the number λ depends on the position r. Therefore, any direction tangent to the concentric circle is the principal direction.

In this analysis, all other sliding directions $\mathbf{v}$ do not satisfy eqn (62), and they are not principal directions of friction.

If the sliding trajectory is a curve and $\eta_2 \neq 0$, then the gyroscopic component of the vector $\mathbf{t}_\rho$ acts. It is always normal to the sliding direction $\mathbf{v}$. Then, the sliding direction and the friction force are not collinear in the most cases.

Property 5. The non-homogeneous friction properties in the form of concentric circles have an axial symmetry with respect to the axis which passes through the center of the concentric circles. The symmetry with respect to translations reduces to zero element. If the additional friction induced by the path curvature is neglected ($\eta_1 = 0$), then there are radial and concentric circular directions of friction extreme values.

If the transformation of the reference system (or an observer) does not change a course of the friction phenomenon, then this transformation belongs to symmetry transformations. In the mathematical sense, the symmetry transformations form a group. Using symmetry properties we can distinguish different types of constitutive models.

For $\eta_2 = \text{const}$ the analysed friction has the axial symmetry, and its group of symmetry G contains rotations about a normal to the contact crossing the center C of the concentric circles

$$\{\mathbf{R}_c^\phi, \phi \in \langle 0, 2\pi \rangle\} \in G. \tag{71}$$

Radial and concentric circular directions have mirror reflections with respect to planes crossing the center of the concentric circles and they belong to elements of the symmetry group $\{\mathbf{R}_c^\phi\}$. There are no mirror reflections with respect to other planes normal to the contact surface. Here, the inversion is identical with the rotation through the angle $\phi = \pi$, and the identity is equivalent to the rotation angle $\phi = 0$.

If the given translation in the contact area does not change the friction phenomenon, then this translation belongs to symmetry transformations. For example, the anisotropic friction homogeneous in the contact area has a continuous symmetry group of translations along arbitrary axes in the space R^2, i.e.

$$\{\mathbf{T}^\delta, \delta \in \langle 0, \infty \rangle\} \in G, \tag{72}$$

where $\mathbf{T}^\delta$ is a translation vector.

The independent variable $\mathbf{n}/r$ of the friction equation depends on the position r. Therefore, there is no symmetry with respect to the translation δ, since $\mathbf{n}/r \neq \mathbf{n}/(r+\delta)$. It acts for any tensor $\mathbf{E}$. In this case, the symmetry group with respect to translations reduces to zero element

$$\{\mathbf{T}^\delta, \delta = 0\} \in G. \tag{73}$$

The analysed friction has not symmetry with respect to translations along any axes.

In experimental investigations it is observed that: (a) friction of wood is dependent on whether

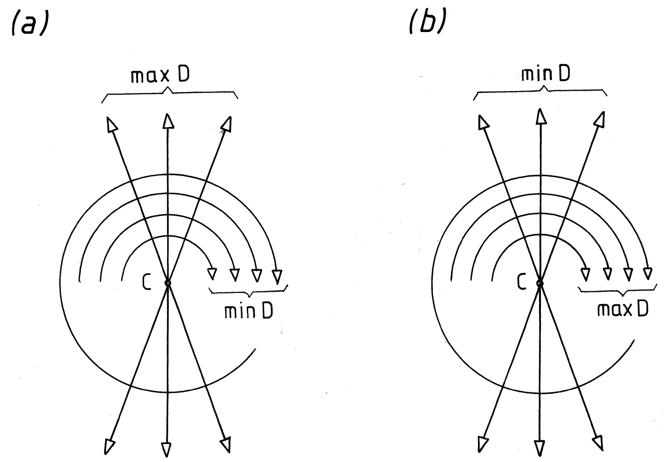

Fig. 4. Radial and concentric circular privileged sliding directions in a plane with non-homogeneous friction properties which form: (a) concentric circles, see Fig. 1 (c); (b) radii relative to the center of the plane, see Fig. 1(d). D is the dissipation function.

sliding takes place parallel to or perpendicular to the wood fibers; (b) when two surfaces of PTFE (Teflon) are oriented with their molecular chains, in parallel, the friction is approximately 30% higher when sliding occurs across the chains than when sliding occurs along them (Tabor and Williams, 1961); (c) friction and wear of composites depend on a fiber orientation with respect to the sliding direction, it has been shown that the lowest coefficient of friction and rate of wear are obtained when fibers are oriented parallel to the direction of sliding (Sung and Suh, 1979; Chang, 1983; Minford and Prewo, 1985; Nayeb-Hashemi et al., 1991; Saka et al., 1992); (d) for surfaces with definite marks (surface texture) friction is smaller when the matching lines lie in the direction of sliding and it reaches the greatest values when they are perpendicular to the sliding direction (Halaunbrenner, 1960; Matalin, 1970; Wang et al., 1992). Generally, it is observed that the sliding 'along the marks' (i.e. wood fibers, molecular chains of PTFE, composite fibers, machining marks) occurs with the lowest resistance to motion, and it has the greatest resistance in the direction 'perpendicular to the marks'.

Let us consider particular properties of the non-homogeneous friction defined with the aid of the tensor $\mathbf{C}$ of the form (21). Here, we neglect the addition friction ($\eta_1 = 0$). In this case, there are two types of privileged sliding directions: the first is radial (i.e. along the radii from the center of concentric circles), the second in concentric circular (i.e. along the concentric circles), see Fig. 1(c) and Fig. 4(a). There is an infinite number of both privileged directions of friction.

As a measure of the lowest and greatest resistance to motion we take a value of energy dissipated in friction process. Let the power of the friction force referred to the unit velocity ($V = 1$) be a dissipation function, i.e.

$$D = -\mathbf{t}\cdot\mathbf{v} \equiv \mu_{\alpha}^{\parallel} N. \tag{74}$$

In the case of radial directions, the sliding takes part ideally 'perpendicular to the marks', and one can expect that the radial direction is a direction of the greatest resistance to motion in the contact. The energy dissipated in friction for the sliding in radial directions achieves the maximum value

$$D_{\max} = \max_{\substack{r\in(0,\infty) \\ \alpha\in\langle 0,2\pi\rangle}} D(r,\alpha) \tag{75}$$

for all contact points i.e. for every circle radius $r\in(0, \infty)$ and for all sliding directions $\alpha\in\langle 0, 2\pi\rangle$, Fig. 4(a). α is a measure of an oriented angle between the reference direction Ox and the sliding direction $\mathbf{v}$ (see Fig. 2).

In the case of rectilinear radial directions the component $\mathbf{t}_\rho$ is always equal to zero. Thus, the component $\mathbf{t}_0$ guarantees the maximum of dissipated energy for the radial directions, and the friction coefficient has the maximum value in these directions

$$\mu_{\max} = \max_{\substack{r\in(0,\infty) \\ \alpha\in\langle 0,2\pi\rangle}} \mu_\alpha^{\|}(r,\alpha). \tag{76}$$

We assume that $\mu_2 = \mu_{\max}$.

Particular friction properties of the concentric circular directions are associated with the sliding along the concentric circles, i.e. 'along the marks'. Then, the sliding has the lowest resistance to motion, and it does not depend on a position in the circle (i.e. it is valid for any sliding velocity unit vector tangent to the circle).

In the sliding along the concentric circles, the constraint imposed on the sliding and expressed by the coefficient η_2 is active. Thus, the inclination angle β exists, and its value depends on the coefficient η_2 and on the radius r.

It is assumed that $\eta_1 = 0$, then the component $\mathbf{t}_\rho$ is normal to the circular trajectory, and the component $\mathbf{t}_0$ guarantees the minimum of dissipated energy. For the sliding along the concentric circles the energy dissipated in friction achieves the minimum value

$$D_{\min} = \min_{\substack{r\in(0,\infty) \\ \alpha\in\langle 0,2\pi\rangle}} D(r,\alpha) \tag{77}$$

for every point r of the contact and for all sliding directions α, Fig. 4(a). The minimum value of the dissipated energy does not depend on the position of the contact point with respect to the center of concentric circles. The coefficient $\mu_\alpha^{\|}$ of the friction force component tangent to the concentric circular trajectory is constant for $\forall r\in(0, \infty)$. Then, the friction coefficient has the minimum value in the sliding along the concentric circles

$$\mu_{\min} = \min_{\substack{r\in(0,\infty) \\ \alpha\in\langle 0,2\pi\rangle}} \mu_\alpha^{\|}(r,\alpha) \tag{78}$$

We denote that $\mu_1 = \mu_{\min}$. Taking $\eta_1 = 0$, all concentric circular sliding trajectories are equivalent. With the aid of $\eta_1 \neq 0$, we can distinguish different concentric circular trajectories.

Notice that in an infinity a direction tangent to the concentric circle is not equivalent to the adequate radial direction. The radial directions are always 'perpendicular to the marks' and the tangent direction in an infinity is 'along the marks', Fig. 5.

Let us assume that the coefficient η_2 in eqn (23) is constant, then the inclination angle β, the friction force value $|\mathbf{t}|$ and the coefficients μ_α and $\mu_\alpha^{\perp}$ are constant for all points of the circular trajectory. Hence, the main features of the concentric circular directions are following: the energy

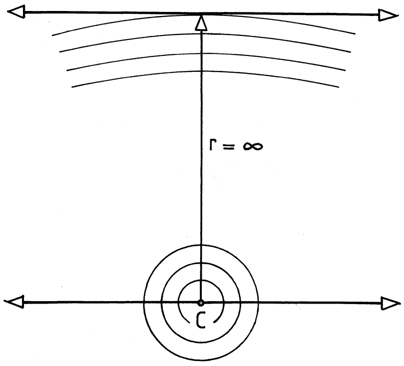

Fig. 5. In an infinity a direction tangent to the concentric circle is not equivalent to the adequate radial direction.

dissipated in friction has the minimum value, the inclination angle $\beta \neq 0$ depends on the circle radius r (it is constant for all points of the circle) and $\beta \to 0$ for $r \to \infty$. Other values of limits are as follows

$$r \to \infty \begin{cases} \mathbf{t}_\rho \to \mathbf{0} & \text{(i.e. } \beta \to 0) \\ \mathbf{t} \to \mathbf{t}_0 & \text{(i.e. } \mu_\alpha(\mathbf{t}) \to \mu_\alpha(\mathbf{t}_0)) \end{cases}. \tag{79}$$

Next, let us assume that the coefficient η_2 in eqn (23) depends on the sliding velocity V. Then, the inclination angle β, the friction force value $|\mathbf{t}|$ and the coefficients μ_α and $\mu_\alpha^{\perp}$ depend on the velocity V at the given point of the circular trajectory. In this case, the angle β function depends on the position r, the sliding direction α and the sliding velocity V, i.e.

$$\beta = \beta(r, \alpha, V), \tag{80}$$

where, $r \in (0, \infty)$, $\alpha \in \langle 0, 2\pi \rangle$, $V \in < 0,\ \infty)$. The function (80) has the following limit value with respect to the circle radius r

$$\lim_{r \to \infty} \beta(r, \alpha, V) = 0, \quad \forall \alpha \in \langle 0, 2\pi \rangle, \quad \forall V \in < 0, \infty). \tag{81}$$

In this case, the friction coefficient μ_α function (17) of the form

$$\mu_\alpha(\mathbf{t}) = \mu_\alpha(r, \alpha, V), \tag{82}$$

has the following limit value

$$\lim_{r \to \infty} \mu_\alpha(r, \alpha, V) = \mu_\alpha^{\parallel} = \mu_{\min}, \quad \forall \alpha \in \langle 0, 2\pi \rangle, \quad \forall V \in < 0, \infty). \tag{83}$$

Functions of μ_α and β depend on the position r of the contact point with respect to the center of concentric circles. Hence, the friction force value is a local quantity. The friction force value changes from point to point, and some properties of friction vanish in an infinity ($r \to \infty$). However, particular properties of friction (radial and concentric circular directions of extreme friction values) refer to global properties of the non-homogeneous friction in the form of concentric circles.

Non-homogeneous friction properties which form a series of radii relative to the center of the contact surface Fig. 1(d) have two types of privileged sliding directions, i.e. radial and concentric circular, see Fig. 4(b). The sliding in the radial direction takes part 'along the marks' i.e. with the lowest resistance to motion. The sliding along the concentric circles takes part 'perpendicular to the marks' i.e. with the greatest resistance to motion. Therefore, one can define that particular friction in the frame of this study taking into account $\mu_1 = \mu_{\max}$ and $\mu_2 = \mu_{\min}$ (i.e. by changing the indices of the friction coefficients corresponding to the friction extreme values).

An isotropic tensor $\mathbf{C}$ and the tensor $\mathbf{E}$ of the form (22) define isotropic non-homogeneous friction. In that case, friction depends on the position with respect to a singular point of the contact area but there are no privileged sliding directions. It means, at some instances, sliding 'along the marks' and 'perpendicular to the marks' can occur with the same resistance to motion, then $\mu_1 = \mu_2 \equiv \mu$, $\mathbf{C} = \mu\mathbf{1}$.

Property 6. The tensors $\mathbf{C}$ and $\mathbf{E}$ have orthotropic and anisotropic properties, respectively.

The subgroup $G(\mathbf{C})$ of the full orthogonal group O is a symmetry group of the friction tensor $\mathbf{C}$, if it satisfies the following

$$G(\mathbf{C}) = \left\{\mathbf{R} : \mathbf{R} \in O, \quad \left(\overset{2}{\underset{1}{\otimes}} \mathbf{R}\right) \cdot \mathbf{C} = \mathbf{C}\right\}, \tag{84}$$

where, the linear operator $\overset{2}{\underset{1}{\otimes}}$ denotes two compositions (contractions) of the following form

$$\left(\overset{2}{\underset{1}{\otimes}} \mathbf{R} \cdot \mathbf{C}\right) = C^{ij}\mathbf{R}\mathbf{k}_i \otimes \mathbf{R}\mathbf{k}_j = \mathbf{R}(C^{ij}\mathbf{k}_i \otimes \mathbf{k}_j)\mathbf{R}^T = \mathbf{R}\mathbf{C}\mathbf{R}^T \quad i,j = 1,2. \tag{85}$$

This rule defines the tensor symmetry with respect to the transformation $\mathbf{R}$. Here, we employ the identity ($\mathbf{1}$), the inversion ($-\mathbf{1}$) and mirror reflections ($\mathbf{J}_{m_i}$) as symmetry operations.

The mirror reflection $\mathbf{J}_{m_i}$ means a reflection in a plane normal to the $\mathbf{m}_i$-axis, and it can be defined by a composition of the rotation about $\mathbf{m}_i$ and the central inversion ($-\mathbf{1}$), i.e.

$$\mathbf{J}_{m_i} = -\mathbf{1}\mathbf{R}^{\pi}_{m_i}, \tag{86}$$

where, $\mathbf{R}^{\pi}_{m_i}$ describes the rotation about the axis $\mathbf{m}_i$ through the angle of rotation π.

The tensor $\mathbf{C}$ of the form (21) is an orthotropic tensor. Coefficients μ_1 and μ_2 are eigenvalues of the tensor $\mathbf{C}$, and two orthogonal eigenvectors $\mathbf{m}_1$ and $\mathbf{m}_2$ coincide with the unit vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ and they satisfy the following equation

$$\mathbf{C}\mathbf{k}_i = \mu_i\mathbf{k}_i, \quad i = 1,2 \tag{87}$$

where, $\mathbf{k}_i \cdot \mathbf{k}_j = \delta_{ij}$, $i, j = 1, 2$. Therefore, the tensor (21) is written with respect to the tensor basis composed of eigenvectors (so-called spectral decomposition of the tensor).

The symmetry group of the orthotropic tensor has the trivial subgroup $\pm\mathbf{1}$ and the subgroup of mirror reflections with respect to two planes orthogonal to the eigenvectors $\mathbf{k}_1$ and $\mathbf{k}_2$, i.e.

$$G(\mathbf{C}) = \{\pm\mathbf{1}, \mathbf{J}_{k_1}, \mathbf{J}_{k_2}\}. \tag{88}$$

In general, any second-order symmetric tensor has the symmetry group of the type (88), and it satisfies the following condition

$$\mathbf{C} = \mathbf{C}^T. \tag{89}$$

The second-order tensor is symmetric if its representation matrix is symmetric. The tensor **C** representations (21) and (27) satisfy these conditions.

The tensor **E** of the form (22) is an anisotropic tensor. The anisotropic tensor has a trivial two-element group of symmetry

$$G(\mathbf{E}) = \{\pm\mathbf{1}\}. \tag{90}$$

There are no restrictions on the anisotropic tensor representation.

The tensors **C** and **E** are defined locally i.e. at the given point of the contact surface (**C**) and at the given point of the sliding trajectory (**E**). Since the description deals with non-homogeneous friction, therefore local symmetry properties of the tensors **C** and **E** are completely different from global properties of the considered friction. This is the reason that the global axial symmetry (i.e. $\{\mathbf{R}_c^\phi\}$) and the translation symmetry (i.e. $\{\mathbf{T}^\delta, \delta = 0\}$) do not restrict forms of the tensors **C** and **E**.

Property 7. In the case of non-homogeneous friction properties in the form of concentric circles, we can distinguish positive and negative additional friction taking into account a sign of the coefficient η_1.

Taking tensors **C** and **E** of the form (21) and (22) and the relations (67), we obtain from (12) the following friction force for the sliding along concentric circles ($\rho = r$)

$$\mathbf{t} = -N\left[\left(\mu_1 + \frac{\eta_1}{r}\right)\mathbf{v} + \frac{\eta_2}{r}\mathbf{n}\right]. \tag{91}$$

Here, the coefficient of the friction force component collinear with the sliding direction **v** is given by

$$\mu_\alpha^\parallel = \mu_1 + \frac{\eta_1}{r}, \tag{92}$$

where, $\mu_1 > 0$, $\eta_1 \in R$. In the case of sliding along concentric circles $r \in R^+$, the coefficient $\mu_\alpha^\parallel$ as a function of the position r has two different courses depending on a sign of the coefficient η_1, see Fig. 6 ($\mu_1 = 0.1, \eta_1 = \pm 0.01m, r \in (0, 1.0m>$). If $\eta_1 > 0$ (positive additional friction), then friction decreases when r increases. If $\eta_1 < 0$ (negative additional friction), then friction increases when r increases. A limiting radius (i.e. a lower limit of radii r) exists in the case $\eta_1 < 0$, and it is equal to $r_l = \eta_1/\mu_1 = 0.1$ m.

An infinite friction in $r = 0$ and negative values of friction in the domain $r \in \langle 0, r_l \rangle$ in the case $\eta_1 < 0$ must be excluded from the considerations, since they are physically meaningless.

In the case of $\eta_1 < 0$, maximum friction is for large radius of the circle and significant reduction in friction is when the radius approaches the limiting radius r_l. Therefore, the negative additional friction ($\eta_1 < 0$) leads to the results which coincide with experimental observations carried out by Briscoe and Stolarksi (1979, 1985, 1991).

Property 8. A value of the friction force at the given point of the contact area depends on the sliding direction and on a shape of the sliding path. Considering different sliding trajectories we

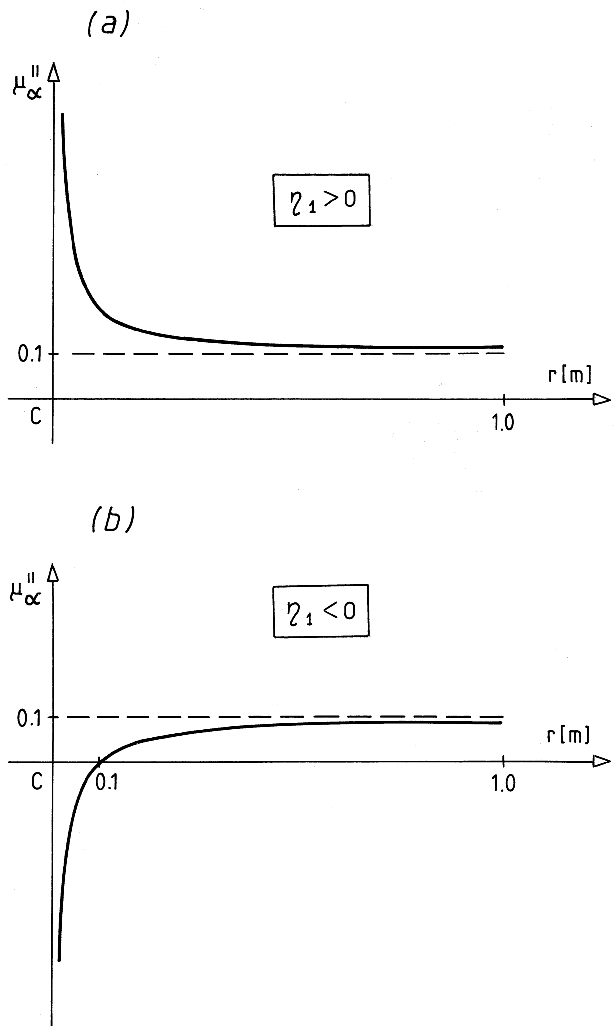

Fig. 6. Coefficient $\mu_\alpha^{\parallel}$ of the friction force component collinear with the sliding direction vs radius r, in two cases: (a) positive additional friction; (b) negative additional friction.

obtain different friction force hodographs at the given contact point. Furthermore, the sliding initiated by an applied force or velocity depends on the trajectory.

A curve drawn by the friction force vectors attached to the origin of the coordinate system Ot^1t^2 is called the hodograph of the friction force. t^1 and t^2 are components of the friction force with respect to the reference system. Sometimes it is called a cross-section of a friction cone. In the case of classical friction law, a shape of the hodograph curve can be obtained by finding friction forces in all sliding directions.

The shape of the curve drawn by the vector $\mathbf{t}_0$ can be found using the identity relation $\mathbf{v} \cdot \mathbf{v} = 1$, after substitution

$$\mathbf{v} = -\frac{1}{N}\mathbf{C}^{-1}\mathbf{t}_0, \tag{93}$$

which follows from the definition (14). Hence, the friction vector $\mathbf{t}_0$ draws an ellipse defined by the following equation

$$\mathbf{t}_0^T\mathbf{C}^{-T}\mathbf{C}^{-1}\mathbf{t}_0 = N^2. \tag{94}$$

Taking into account tensor $\mathbf{C}$ of the form (21), we obtain the friction hodograph equation given by

$$\frac{(t^1)^2}{(\mu_1)^2} + \frac{(t^2)^2}{(\mu_2)^2} = N^2. \tag{95}$$

Here, the components t^1 and t^2 of the vector $\mathbf{t}_0$ are given with respect to the basis $\{\mathbf{k}_1, \mathbf{k}_2\}$. Half-axes of the ellipse are equal to $\mu_1 N$ and $\mu_2 N$. At any point of the contact area, the half-axis $\mu_1 N$ is tangent to the concentric circle and the half-axis $\mu_2 N$ is directed along radii of the concentric circles.

In the case of isotropy ($\mu_1 = \mu_2 \equiv \mu$), the hodograph equation reduces to the following circle equation

$$(t^1)^2 + (t^2)^2 = \mu^2 N^2. \tag{96}$$

If we take into account the dissipative component of the vector $\mathbf{t}_\rho$, then the curve drawn by the vector $\mathbf{t} = \mathbf{t}_0 + \mathbf{t}_\rho$ is not an ellipse. The dissipative component of the vector $\mathbf{t}_\rho$ causes, that the friction hodograph depends on the sliding path curvature. The friction force has different values for various sliding path curvatures.

There is no possibility to define friction hodograph without a knowledge of sliding path shapes. We can do that in some particular cases. Assuming that at the given contact point the sliding trajectories for all sliding directions $\mathbf{v}$ are straight lines ($\rho = \infty$), then the hodograph equation reduces to the cases (94) and (95). One can consider the sliding trajectories of the given type constant for all sliding directions $\mathbf{v}$, e.g. circles of the given radius r. Then the ellipse drawn by the vector $\mathbf{t}_0$ is modified by the dissipative component $-(N\eta_1/r)\mathbf{v}$ added to $\mathbf{t}_0$ and constant for all sliding directions. Another approach—different sliding trajectories can be taken for various sliding directions $\mathbf{v}$, e.g. a circle of the given radius for the direction tangent to the concentric circle and a straight line for the radial direction with respect to the concentric circles. For all intermediate directions $\mathbf{v}$ the sliding trajectories can change smoothly from the circular to the straight line. The sliding direction $\mathbf{v}$ is always tangent to the actual trajectory. Then the ellipse of the vector $\mathbf{t}_0$ is modified by the component $-(N\eta_1/\rho)\mathbf{v}$ which changes its value depending on the sliding direction.

If the sliding is caused by an initial velocity, then an initial sliding direction coincides with the velocity direction. The initial velocity vectors defines unit vectors $\mathbf{v}$ and $\mathbf{n}$, and the sliding trajectory can be assumed to be a straight line ($\rho = \infty$) at the initial instant.

An initial sliding direction $\mathbf{v}$ caused by an applied force $\mathbf{f}$ is such that the sum of the components orthogonal to it (the applied force and the friction force) is equal to zero, i.e.

$$[\mathbf{f} - (\mathbf{f}\cdot\mathbf{v})\mathbf{v}] + [\mathbf{t} - (\mathbf{t}\cdot\mathbf{v})\mathbf{v}] = \mathbf{0}. \tag{97}$$

Then, the absolute value of the applied force projection on the sliding direction is greater then the absolute value of the friction force projection on this direction

$$|\mathbf{f} \cdot \mathbf{v}| > |\mathbf{t} \cdot \mathbf{v}|. \tag{98}$$

The initial sliding direction is the direction of the lowest resistance to motion. This definition is identical with the definition in the case of classical friction law.

Let us consider the initial sliding in the case of the non-homogeneous friction in the form of concentric circles. If the additional friction is positive ($\eta_1 > 0$), then a straight line is the most advantageous initial sliding trajectory. In that case, the dissipative and gyroscopic components of the vector $\mathbf{t}_\rho$ are equal to zero, and the applied force $\mathbf{f}$ causing the sliding is the lowest between all admissible forces, i.e. the straight line trajectory guarantees the lowest resistance to motion.

In the case of negative additional friction ($\eta_1 < 0$), a circle of the limiting radius $r = r_l$ is the most advantageous initial sliding trajectory. In that case, friction reduces to zero, i.e. $\mu_\alpha^\| = 0$, and there is no resistance to sliding.

4. Conclusions

(1) A non-homogeneous anisotropic friction constitutive equation with the sliding path curvature effects is presented. The equation and its variables satisfy the axiom of objectivity. The condition of dissipated energy (the Second Law of Thermodynamics) restricts components of the tensors in the equation.
(2) In the surface with non-homogeneous friction properties in the from of concentric circles, besides rectilinear (radial) also curved (concentric circular) privileged sliding directions occur. Principal directions of friction and extreme friction directions are distinguished, taking into account restrictions on coefficients η_1 and η_2.
(3) Non-homogeneous friction in the form of concentric circles has the global axial symmetry. Its group of symmetry with respect to translations reduces to zero element. The constitutive equation tensors $\mathbf{C}$ and $\mathbf{E}$ are defined locally and they have orthotropic and anisotropic properties, respectively.
(4) Positive additional friction and negative additional friction are induced by the sliding path curvature. The friction force hodograph and the sliding initiated by an applied force or velocity depend on the sliding trajectory.
(5) The phenomenological friction equation has two main advantages: it contains a finite number of parameters (it is easy to plan simple experiments), it can be expanded to other non-homogeneous anisotropic friction types. Considerations presented in this study require further experimental work and test evaluation.

Most of the activity in anisotropic friction investigations is motivated by the interest in fibrous and laminated composites and in ceramics applied in modern machine component parts operating in contact conditions. A large number of composite materials have good wear properties and can be used unlubricated. Ceramics are attractive as mechanical components because of their extreme hardness and excellent wear resistance.

A detailed analysis of non-homogeneous anisotropic friction would be useful amongst engineers and those engaged in machining operations and in applications of surfaces with textured profiles. Surfaces of the prescribed regular microgeometry (parallel or spiral microgrooves, etc.) distinguish a great resistance to seizing and to abrasion in the conditions of dry friction. The microgrooves can improve lubrication.

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